Cross-Gradient Joint 3D Inversion of Geophysical Data with Applications to Gravity and Magnetics

E. Fregoso-Becerra and L. A. Gallardo
Earth Science Division, CICESE, Mexico (fregosob@cicese.mx)

Summary
We extend the cross-gradients philosophy for joint inversion to three-dimensional environments and developed a solution procedure based on a statistical formulation and singular value decomposition. We apply the proposed solution to the joint 3D inversion of gravity and magnetic data where we gauge the advantages of this new formulation on comparative experiments. We found that, compared to the separate inversion of gravity and magnetic data, the ambiguities of each data set are reduced by obtaining models that enhance the structural similarities.

Introduction
Our understanding of complex subsurface processes occurring in many geological environments demands a detailed analysis of the distribution of several of their physical properties. Although some geological environments may be represented by two-dimensional or even one-dimensional structures, a detailed analysis necessarily relies on an accurate determination of three-dimensional models.

By relying on direct or indirect parameter interdependence, the joint inversion can successfully restrict the model space to only those models that satisfy some cross-linked characteristics. In this case, the selection of such links is a crucial step in joint inversion and this has derived on diverse philosophies (e.g. Gallardo and Meju, 2003, 2004; Haber and Oldenburg, 1997; Saunders et al., 2005; Bosch and McGaughey, 2001; Zhang and Morgan, 1996). An emerging philosophy relies on the idea that physical properties tend to change at the same location and focuses on the search of structural similarities.

In this work, we extended the cross-gradient technique, proposed by Gallardo and Meju (2003; 2004) for the two dimensional case, to 3D structures. We formulate an objective function based on equality constraints and solved the 3D joint inverse problem under the Generalized Non Linear Least Squares framework developed by Tarantola and Valette (1982). We proved our formulation on a synthetic experiment using gravity and magnetic data, and compared the results of separate and joint 3D inversions. The results clearly show the advantages of jointly inverting gravity and magnetic data in heterogeneous three-dimensional environments.

3D Cross-Gradient Joint Inversion Formulation
A cross-gradient joint 3D inversion problem of two geophysical data can be reduced to the search of two 3D physical models that being structurally similar, satisfy both geophysical data. Although the formulation described in Gallardo and Meju (2003) seems complete to generalize a joint 3D inversion procedure, we prefer to base our inversion algorithm on the generalized least-squares formulation proposed by Tarantola and Valette (1982). We expect that, unlike the Lagrange multiplier solution developed by Gallardo and Meju (2003, 2004), this formulation will provide a more robust statistical framework necessary to quantify the advantages that the cross-gradient constraint brings into a joint 3D problem. In a broad sense, the approach of Tarantola and Valette (1982) can incorporate conventional regularizing constraints such as Tikhonov regularization, as new random variables i.e. as
additional non-geophysical information. However, the incorporation of the cross-gradient constraint for 3D case, given by

$$ t(x, y, z) = \nabla m_i(x, y, z) \times \nabla m_j(x, y, z) = 0, $$

involves subtle differences.

Based on the nonlinear generalized least-squares formulation of Tarantola and Valette (1982), we found the following iterative solution:

$$ \hat{m}_{i+1} = \hat{m}_i + N_i^{-1} \left[ G_i^T \cdot C_{dld}^{-1} \cdot \left( d_0 - g_d(\hat{m}_i) - C_{m0}^{-1} \cdot (\hat{m}_i - m_0) \right) \right] 
- N_i^{-1} \cdot B_i \cdot N_i^{-1} \cdot B_i^T \cdot \left[ G_i^T \cdot C_{dld}^{-1} \cdot \left( d_0 - g_d(\hat{m}_i) - C_{m0}^{-1} \cdot (\hat{m}_i - m_0) \right) \right] + g_m(\hat{m}_i) \right\} \right] $$

where

$$ N_i^{-1} = \left( G_i^T \cdot C_{dld}^{-1} \cdot G_i + C_{m0}^{-1} \right)^{-1}. $$

Here, $\hat{m}$ and $m_0$ are the initial and the a priori models, respectively; $C_{dld}$ corresponds to covariance matrices of the joint data set and regularization terms; $C_{m0}$ is the covariance matrices for the a priori model parameters; $G$ is a partitioned matrix that contains both sensitivity matrices as well as Laplacian derivative matrices; $B$ is the Jacobian matrix of $t$; $d_0 - g_d(\hat{m}_i)$ includes the data misfits and regularizing terms and $g_m(\hat{m}_i)$ is the cross-gradient vector. Note that while $N_i$ is well posed, $B_i N_i^{-1} B_i^T$ is not a full rank matrix, and, it is inverted using singular value decomposition (SVD).

### 3D Joint Inversion Of Gravity and Magnetic Data

We implemented the inversion formulation to the joint 3D inversion of gravity and magnetic data. For this, we compose our subsurface model as an aggregate of rectangular prisms with homogeneous density and magnetization, and compute its gravity and magnetic responses using the equations developed by Bhaskara-Rao et al. (1990) and Bhattacharyya (1966).

### Synthetic test model

To prove our 3D cross-gradients joint inversion algorithm, we considered an aggregate of 512 prisms in a volume that is 80 m in both horizontal directions and depth. Using this model we set a cubic heterogeneity which is embedded in a homogeneous media of density and induced magnetization equal to 0 (see Figure 1). The cube is 20 m by side and it is buried at a depth of 30 m. The density of the cube is 1.0 g/cm$^3$ and the induced magnetization is 1.0 Ampere/m.

A total of 1681 2m-spaced data were calculated on the surface for both gravity and magnetic fields. The data were added random noise with standard deviation of 2% of the maximum amplitude of their respective anomalies.

### Separate vs. joint inversion of test data

In order to gauge the main advantages of the joint 3D inversion formulation, we perform two comparative inversion experiments. First, the gravity and magnetic data were inverted separately; then, the same data were inverted jointly, incorporating cross-gradients constraints.

For the process, we stated the a priori and initial models as null homogeneous models and, we stated their appropriate covariance matrices with large values. We used the formulation [2] and [3] applying several smoothing factors until satisfactory misfit and convergence (after six iterations) of the process was achieved.

Figures 2 and 3 show the density and magnetization models obtained from the separate and joint inversion, respectively. By comparing these figures it is clear that the models obtained from the joint
inversion show several improvements. For instance, the bottom of the cube is better defined in its original position in depth, and, the broad tails that appeared in the separately estimated models, where there is no causative body, are reduced in their value and extent.

**Model assessment**

We found that both, joint and separate inversion experiments achieved a satisfactory data misfit (0.9961 for the gravity and 0.9556 for the magnetic data). While these data misfits account for the geophysical support of the models, the structural similarity achieved by the models can be described by the cross-gradient values. For this, we plotted each one of the components of the cross-gradient vector function (Equation [1]) for both pairs of models (Figures 4 and 5). Each one of these components quantifies the structural similarities in perpendicular planes to the component direction. For example, the $t_y$ component (Figure 4b) quantifies the structural similarity in the x-z plane by observing spatial variations in x and z directions. For this experiment, the largest cross-gradient absolute values occur at the bottom of the model domain (see Figure 4), and correlate to the proper geophysical data resolution. Regions with major structural concordance correlate to homogeneous portions of either model or to regions with parallel changes.

Note that, the numerical values of the cross-gradient function (Figure 5) for the jointly inverted models are six orders of magnitude smaller than those of the separately estimated models, indicating an increased structural similarity.

![Figure 1](image1.png)  
**Figure 1.** Three-dimensional model of a buried cube in a homogeneous medium. The location and dimensions of the cube are indicated by its projections on the coordinate planes.

![Figure 2](image2.png)  
**Figure 2.** Three-dimensional view of the density (a) and magnetization (b) models that resulted from the separate inversion of the gravity and magnetic data.
Conclusions

We have developed an iterative formulation for the cross-gradient joint 3D inversion of two geophysical data, which we have applied to gravity and magnetic data. The developed formulation has the advantage of involving linear and no linear functional relationships between both data and parameters and gives a solid statistical support for model appraisal. The formulation, as applied to potential methods, shows that the cross-gradient joint inversion technique succeeds on finding geophysically supported models with major structural concordance between them. The results suggest that jointly inverted models show better definition of structural features partially overcoming the inherent lack of depth resolution. However, the results also show that whereas the cross-gradient constraint produces structurally similar models, the actual numerical value of the parameters largely depends on the geophysical data themselves. As a result, it can be expected that a the incorporation of complementary information such as that provided by other geophysical techniques or a priori models, could produce improved models.

References


